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Some characteristics of the free surface in the wedge entry problem

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SUMMARY

The classical hydrodynamics problem of the constant-velocity entry of a prismatic wedge into a weightless incompressible inviscid fluid is investigated, and some newly-derived characteristics of the free surface are presented. The line integration along the free surface, which is an essential part of the solution method utilising Wagner's function, is shown to possess a direction-dependent stability. Expressions are developed for the far-field flow characteristics which take into account the horizontal velocity and which avoid the other approximations of previous methods. Finally, the nature of the spray-tip singularity is investigated.

1. Introduction

The response of a free surface to the impact of a solid body is one of the most challenging problems in hydrodynamics. Numerous versions of the problem have been treated using various methods and combinations of simplifying assumptions in an effort to deduce the fundamental characteristics of the free-surface response. One of the best-known class of problems is that in which the flow is self-similar, that is, the wetted portion of the body remains geometrically similar in shape and grows in physical size at a rate proportional to t^n, where t indicates time measured from the first instant of contact, and n is an integer. Mathematically, a self-similar flow is defined as one having a velocity potential $\phi(X, Y, Z, t)$ which satisfies the homogeneity relation

$$\phi(X, Y, Z, t) = C^2 t^{2n-1} \phi(x, y, z) \tag{1.1}$$

in which X, Y and Z are the physical cartesian coordinates, C is a constant of dimension length/(time)^n, and

$$x = \frac{X}{Ct^n}; \quad .y = \frac{Y}{Ct^n}; \quad z = \frac{Z}{Ct^n}. \tag{1.2}$$

Thus, in the transformed (x, y, z) space the problem reduces to a steady-state boundary-value problem. If the problem is two-dimensional or axisymmetric, the methods of complex analysis can be applied.

In general, the above condition translates to the following physical requirements: (i) the disturbance must begin from a single point; (ii) the fluid must be incompressible, inviscid and (except for $n=2$) weightless; (iii) all solid boundaries must be rectilinear (at least in the direction of flow). Examples of self-similar flows are:

(a) impact of a prismatic wedge-shaped solid against a wedge of water at a velocity $C = t^{n-1}$. Neither of the wedges need be symmetric, but the angle between their respective faces must be less than π; see Figure 1(a);
(b) the normal impact of a symmetric solid cone against a symmetric cone of fluid;
(c) other shapes which maintain geometric similarity, such as that shown in Figure 1(b). These could not represent solid bodies, but they might be used, for example, to model the first instants of a surface explosion.

To date, solutions have only been achieved for $n=1$ (that is, constant impact speed) and the present work deals with two-dimensional wedge flow with $n=1$.

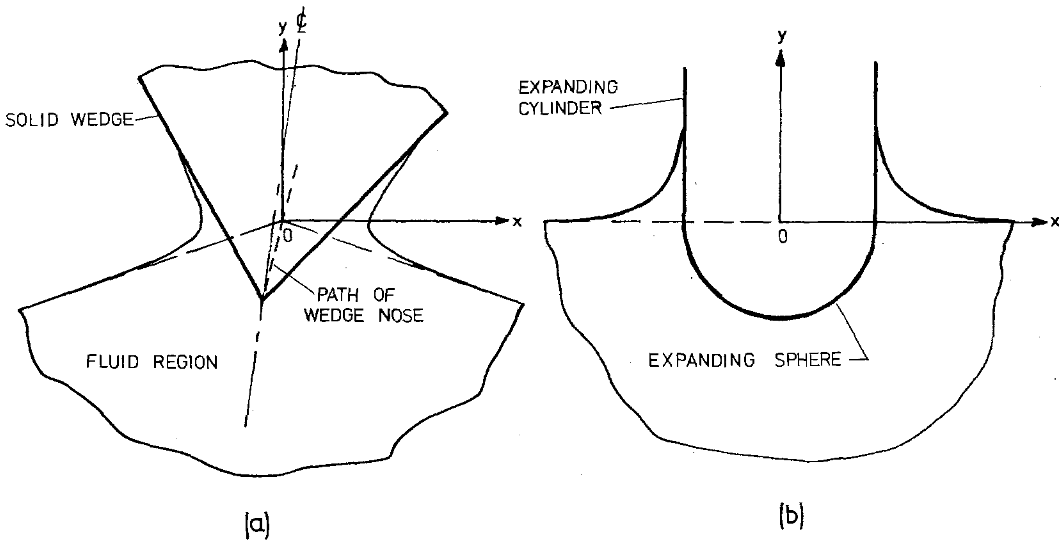

Figure 1. Examples of self-similar flows.

Obviously, in the majority of applications the fluid region is a half-plane, corresponding to the horizontal free surface. If, in addition, the wedge is symmetric and normal to the free surface, example (a) becomes the well-known problem originally formulated by Wagner [14] and illustrated in Figure 2. Since Wagner's pioneer paper, over one hundred publications have appeared dealing with this problem. This large number is due mainly to the fact that, in spite of its idealised nature, the problem has a number of important applications: planing boats, seaplanes, and other high-speed surface craft; "bow-flare" impact of ships, and other cases of high-speed water entry of solid bodies. However, in spite of the large number of attempts, only a few solutions have been achieved. These have generally been of two types: (i) numerical solutions due to Garabedian [7], [8], Borg [2] and Dobrovolskaya [4]; (ii) a conformal

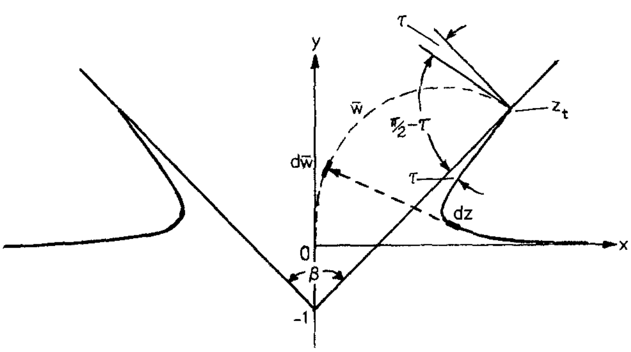

Figure 2. Symmetric wedge entry.

mapping solution due to Hughes [10]. The conformal mapping approach provides information about the entire flow field, and the latter solution revealed several fundamental aspects of free-surface response. However, in the published version of the solution, space limitations made it necessary to omit some of these results and the purpose of this article is to present this additional information. The results can be applied to other instances of self-similar flow. The analysis begins in the next section with a brief summary of the formulation and method of solution of the Wagner problem showing that the solution involves a line integration along the free surface. In Section 3 the stability of this integration is shown to be direction-dependent, requiring that the integration be done in an inwards direction. This in turn requires determining the far-field flow characteristics, in order to calculate the starting values for this integration.

These characteristics are derived in Section 4. Finally, Section 5 investigates the nature of the spray-tip singularity.

2. Formulation and solution of the Wagner problem

The transformation given in (1.2) reduces a self-similar problem to a steady-state boundary-value problem in which all of the boundaries, including the free-surface curve $y=h(x)$, are fixed in position. The boundary conditions are similarly transformed and for the Wagner problem the constant-pressure condition along the free surface becomes:

$$\frac{p}{\rho C^2} = -\phi+xu+yv - \frac{u^2+v^2}{2}$$
$$= \text{constant along } y=h(x) \tag{2.1}$$

in which $u=\partial\phi/\partial x$ and $v=\partial\phi/\partial y$.

The principal difficulty of the problem is that the location of the free surface is unknown and must be found as part of the solution. Since the fluid is ideal and initially undisturbed, the flow is irrotational, and hence the free surface always consists of the same particles. This gives rise to the kinematic condition that the rate of change of the free-surface height is equal to the upwards velocity of fluid particles on the free surface. Wagner showed that this condition results in the following expression for the slope of the free surface:

$$\frac{dh}{dx} = \tan\theta = \frac{y-v}{x-u}. \tag{2.2}$$

In order to use the methods of complex analysis, we define $z=x+iy$ as the complex flow plane and $w=u-iv$ as the complex velocity. (2.2) can then be rewritten as

$$\tan\theta = \frac{\operatorname{Im}(z-\bar{w})}{\operatorname{Re}(z-\bar{w})}$$

or

$$\theta = \arg\left(\frac{dz}{ds}\right) = \arg(z-\bar{w}). \tag{2.3}$$

Wagner also derived a geometric relationship based on the well-known fact that a free surface is always aligned perpendicularly to the local direction of acceleration. In order to illustrate this, the conjugate velocity $\bar{w}$ is superimposed on the z plane in Figure 2. As shown in the figure, a change in velocity $d\bar{w}$ is incurred when moving through a distance dz along the free surface. This change in velocity implies an acceleration and thus the differential element dw must be perpendicular to dz. Wagner therefore introduced the quantity

$$W(z) = \int_{\infty}^{z} (dw\,dz)^{\frac{1}{2}}, \tag{2.4}$$

which we shall refer to as the Wagner function, and showed that because of the perpendicularity of the two differential quantities the image of the free surface in the W plane is a straight line. The other boundaries of the fluid region are straight lines, both in the z plane and the w plane. They will therefore continue to be straight lines when transformed into the W plane, and it can be shown that the resulting fluid region in the W plane is an isosceles (45°) right triangle, as in Figure 3. Dobrovolskaya [4] has demonstrated that this same shape is obtained for all problems having $n=1$, regardless of symmetry or of nose angle of either the fluid wedge or the solid wedge.

Since the shape of the free surface is known in the W plane, W is chosen as the independent variable and other quantities are obtained in terms of W. Hughes [10] defined the complex velocity as a function of W, say $w=\mathscr{L}(W)$, and considered $\mathscr{L}(W)$ as an unknown mapping function which transforms the triangular fluid region of the Wagner plane into the hodograph

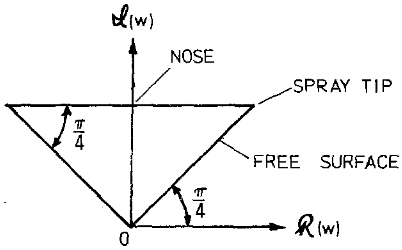

Figure 3. Wagner function plane.

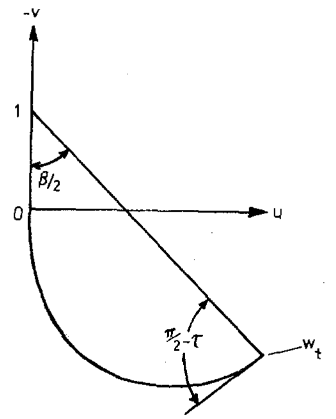

Figure 4. Complex velocity plane (drawn for half wedge only).

plane w (Figure 4). The definition of W in (2.4) can be rearranged to give

$$\frac{dW}{dz} = \mathscr{L}'(W) \tag{2.5}$$

in which the prime denotes differentiation with respect to W. From this it follows that

$$z = \int \frac{dW}{\mathscr{L}'(W)} \tag{2.6}$$

and since the velocity is related to the complex potential ζ by $w = d\zeta/dz$ it is also possible to obtain an integral expression for ζ:

$$\zeta = \int \frac{\mathscr{L}(W)}{\mathscr{L}'(W)}\, dW. \tag{2.7}$$

We thus have a parametric solution for ζ as a function of z, each being expressed in terms of W. In this formulation the principal unknown is the mapping function $\mathscr{L}(W)$, and this must be such as to satisfy all of the boundary conditions. Hughes obtained this function by means of a numerical conformal-mapping technique, but regardless of the method used, the need to satisfy the free-surface boundary condition requires that the location of this curve be determined. This is achieved by integrating the kinematic condition (2.2) which expresses the derivative or slope of the free surface. The integration is done with respect to arc length s along the free surface

$$z = \int e^{i\theta} ds$$

and from (2.3)

$$z = \int e^{i \arg(z-w)} ds$$

or, in terms of W

$$z = \int e^{i \arg(z - \overline{\mathscr{L}(W)})} ds \,. \tag{2.8}$$

3. Directional-dependent stability of free surface integration

The integration implied in (2.8) must be done numerically since z appears on the right-hand side, and it must begin at some point whose position is known. At first sight it would appear immaterial as to which direction is chosen to integrate along the free surface: proceeding outwards from the spray tip or proceeding inwards from some point distant from the wedge. However, if the first direction is chosen, the integration is found to be completely unstable, whereas the second or inwards direction is quite stable. This fact seems to have gone unnoticed in the literature. Several authors have used the above integration technique to determine the free-surface position, but because of the lack of information about conditions near the wedge all of them have fortunately begun at a point distant from the wedge. This unusual "direction-dependent" instability is demonstrated and explained in Table 1, which comprises a systematic qualitative examination of the numerical solution of (2.8) when integration is performed in

TABLE 1

Direction-dependent stability of free surface integration

Portion of free surface with positive slope

Assume that the slope at a given point is *less positive* than the true slope:

Direction	*(x, y) of next position compared with true values*	*Effect on num. and den. of* (2.2)	*Combined effect on next slope*	*Comment*
Inward	y is less + x is more +	num. is more − den. is less −	more +	better
Outward	y is more + x is less +	num. is less − den. is more −	less +	worse

Note:
If the slope had been assumed *more positive*, the effect on both x and y is reversed. Therefore, the combined effect on the next slope is unchanged (i.e. better for inward; worse for outward).

Portion of free surface with negative slope

Assume that the slope at a given point is *less negative* than the true slope:

Direction	*(x, y) of next position compared with true values*	*Effect on num. and den. of* (2.2)	*Combined effect on next slope*	*Comment*
Inward	y is less + x is less +	num. is more − den. is less +	more −	better
Outward	y is more + x is more +	num. is less − den. is more +	less −	worse

Note:
If the slope had been assumed *more negative*, the effect on both x and y is reversed, and thus the combined effect is unchanged.

both the inwards and outwards directions. Regardless of the method employed (predictor-corrector, Runge–Kutta, etc.) the basic procedure always consists of using the slope at a given point to obtain the next point, and continuing from there. Applied to the problem at hand, this basic procedure is:

(a) at a given point, and for a given arc length ds, extrapolate along the slope to obtain the next point (x, y);

(b) insert these new values of x and y into (2.2) (together with values of u and v) in order to evaluate the slope at this next position.

In general, the slope at the given point will be slightly inaccurate due to "discretization" errors. Therefore, the extrapolated point (x, y) has a greater or smaller error here than it did at the given point. As shown in Table 1, the error is always greater for the outward direction and less for the inward direction. Thus, the integration must necessarily begin at some point distant from the wedge and proceed in an inwards direction. This, in turn, requires that expressions be available for the height, velocity, and other free surface quantities at an arbitrary point distant from the wedge. These will now be derived.

4. Calculation of the far-field flow characteristics

It is a well-established fact that a body penetrating a free surface is always a "finite" body, regardless of its shape or size outside the fluid, since only its wetted portion causes disturbance in the fluid. Thus, one way of approximating the wedge flow at all points distant from the wedge is to assume some equivalent finite body for which the flow is known. The approximation will be good or bad depending on the extent to which this assumed flow satisfies the various conditions of the problem. All of the previous efforts to calculate the far-field wedge flow have used this "equivalent-body" technique, following the approach initiated by von Kármán [13]. This involves a number of simplifying assumptions and approximations. Firstly, the flow is assumed to be of impulse type, in which the wedge motion starts instantaneously from rest with the wedge initially submerged at a unit depth. In this case the boundary condition $p=$ constant becomes $\phi=$constant. Also, the boundary is assumed to be a straight horizontal line (that is, the curvature of the free surface is neglected) in order that the equivalent body might possess an axis of symmetry and be more easily represented.

Von Kármán further simplified the problem by using a flat plate as the reflected body. Wagner improved this somewhat by deriving more accurate expressions for both the size and the location of this plate. Bisplinghoff and Doherty [1], Ferdinande [6], and many others have used the diamond-shaped body, which is geometrically more accurate. Fabula [5] used an ellipse so as to avoid the infinite velocity which occurs at the tip of the flat plate and at the corner of the diamond.

Another important limitation of all of these previous analyses is that they take the free-surface velocity to be that along the horizontal axis of symmetry of the body (the line $\phi=0$). Here the velocity is purely vertical and hence this approximation implicitly assumes that the transverse or u-component of the free-surface velocity is zero.

Thus, the equivalent-body approach introduces a number of inaccuracies into the problem, and this may help to explain the relatively limited success which this method has had. An alternative approach is to represent the far-field flow by a complex potential of the form:

$$\zeta = \sum_{m=1}^{\infty} \frac{A_m}{z^m} \tag{4.1}$$

Since the wedge is symmetric and normal to the free surface, the complex coefficients A_m can be expressed as $i^m a_m$, a_m real, and (4.1) becomes

$$\zeta = i\frac{a_1}{z} - \frac{a_2}{z^2} - i\frac{a_3}{z^3} + \frac{a_4}{z^4} + \dots\ . \tag{4.2}$$

By substituting $y=h(x)$ we may obtain the following expressions for the velocity potential and the velocities on the free surface:

$$\phi = \frac{-a_2}{x^2} + \frac{a_1 h(x)}{x^2} + \frac{a_4}{x^4} + O\left(\frac{h(x)}{x^4}\right) + O\left(\frac{1}{x^6}\right) \tag{4.3}$$

$$u = \frac{2a_2}{x^3} - \frac{2a_1 h(x)}{x^3} - \frac{4a_4}{x^5} + O\left(\frac{h(x)}{x^5}\right) + O\left(\frac{1}{x^7}\right) \tag{4.4}$$

$$v = \frac{a_1}{x^2} - \frac{3a_3}{x^4} + O\left(\frac{h(x)}{x^4}\right) + O\left(\frac{1}{x^6}\right). \tag{4.5}$$

These may be substituted into the constant-pressure condition (2.1), in which the constant may be arbitrarily chosen as zero. This yields

$$\frac{3a_2}{x^2} - \frac{2a_1 h(x)}{x^2} - \frac{5a_4 + \frac{1}{2}a_1^2}{x^4} + O\left(\frac{h(x)}{x^4}\right) = 0. \tag{4.6}$$

The equation of the free surface may also be represented by means of a series

$$h(x) = \sum_{n=1}^{\infty} \frac{b_n}{x^n} \tag{4.7}$$

and this must satisfy the kinematic condition (2.2). Substituting the above series expressions for u, v and h into (2.2) and equating terms of like powers gives

$$b_1 = b_3 = 0\ ; \quad b_2 = \tfrac{1}{3}a_1\ ; \quad b_4 = -\tfrac{3}{5}a_3\,.$$

The equation for the free surface is then

$$h(x) = \frac{a_1}{3x^2} - \frac{3a_3}{5x^4} + O\left(\frac{1}{x^6}\right).$$

We note that only the first term will be significant since $h(x)$ only occurs in (4.6) in combination with x^{-2}. Hence we may write

$$h(x) = \frac{a_1}{3x^2} + O\left(\frac{1}{x^4}\right). \tag{4.8}$$

This may now be substituted into (4.6), and since (4.6) must hold for all values of x we may equate to zero each term involving a separate power of x. The result is

$$a_2 = 0 \quad \text{and} \quad a_4 = -\tfrac{7}{30}a_1^2\,. \tag{4.9}$$

Therefore, expressions (4.3), (4.4) and (4.5) become

$$\phi = \frac{a_1^2}{10x^4} + O\left(\frac{1}{x^6}\right) \tag{4.10}$$

$$u = \frac{4a_1^2}{15x^5} + O\left(\frac{1}{x^7}\right) \tag{4.11}$$

$$v = \frac{a_1}{x^2} + O\left(\frac{1}{x^4}\right) \tag{4.12}$$

and these equations describe the far-field free-surface response in the wedge-impact problem. In particular, (4.8), (4.11) and (4.12) provide the required starting values for the inwards integration of (2.8). It is evident that all of the above equations will have good accuracy at large values of x, since the neglected terms are always two orders of magnitude smaller than the retained term.

Obviously, the value of a_1 will depend on the included angle β of the wedge, and in utilizing these equations it is necessary to regard the coefficient a_1 as one of the parameters to be determined in the problem. For instance, Hughes [9] has shown that for $\beta = \pi/2$, $a_1 = 1.09$.

5. Spray tip singularity

Let us denote the angle in the z plane at which the fluid surface contacts the wedge as τ. This quantity has been the subject of much discussion in the literature. Early workers, such as Wagner, intuitively assumed this to be a small but finite angle, and the numerical solutions of Pierson [12] and Borg [2] clearly show this to be the case, but subsequent theoretical analyses have not succeeded in producing an expression for τ in terms of β. Garabedian [8] showed that the angle is bounded by the inequality $0<\tau\leqq\pi/4$. Mackie [11] showed that if the curvature of the free surface at the spray tip is zero, then $\tau<\pi/6$. In the more complete numerical solution of Dobrovolskaya [4] the angle is shown to be small but finite, and this is confirmed by the conformal solution of Hughes [10]. However, until now no author has explained the mathematical character of the spray tip; that is, whether this point is a singularity and, if so, what is the nature of the singularity.

At the spray tip the fluid is moving along the wedge face at a velocity w_t, and it may be shown from similarity that the magnitude of w_t is given by $|w_t|=|z_t|$. Thus, w_t is finite and since w is the derivative $d\zeta/dz$, the $\zeta\to z$ transformation must be conformal at this point. This in turn means that the included angle of the spray tip in the ζ plane is also τ. However, in the case of the w plane the included angle is markedly different, being $\pi/2-\tau$ as shown in Figure 4. This is a direct consequence of the perpendicularity of $d\bar{w}$ and dz demonstrated earlier, which implies that at the spray tip the free surface is perpendicular to the $\bar{w}$ curve, as is shown in Figure 2, and the angle between this curve and the wedge face is therefore $\pi/2-\tau$. This must also be true for w and therefore, as shown in Figure 4, the included angle of the spray tip in the w plane is $\pi/2-\tau$.

We thus have the situation in which the $\zeta\to w$ and $z\to w$ transformations are not conformal at the spray tip, whereas the $\zeta\to z$ transformation is conformal at this point. Therefore, the seemingly complicated relationship between these variables is seen to have a relatively simple explanation: the complex potential $\zeta(z)$ must be such that at this point its first derivative $d\zeta/dz$ (which is w) is finite, while its second derivative is zero. Also, since $\zeta\to z$ is conformal, it follows that w will also have a zero derivative with respect to ζ at this point.

Let us further denote the angle of the spray tip in terms of a parameter α defined by

$$\tau=\alpha\frac{\pi}{2}. \tag{5.1}$$

Since the corresponding angle in the W plane is known to be $\pi/4$, these two planes must at this point bear the proportional relationship

$$z-z_t\propto(W-W_t)^{2\alpha}$$

which could be expressed as an equation by the addition of a complex constant of proportionality. By differentiating this expression and invoking (2.4), it may be shown that at the spray tip the functional relationship between w and W is

$$w-w_t\propto(W-W_t)^{2(1-\alpha)}.$$

For a 90° wedge the conformal mapping solution of Reference [10] gave $\tau=0.04$ so that from (5.1) the angle of the spray tip is $\tau=3.6°$. The numerical solution of Dobrovolskaya [4] gave $\tau=4.5°$, whereas the numerical solution of Cumberbatch [3] to the almost equivalent problem of a 90° water edge and plane wall gave $\tau=3.0°$.

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